



Graphs

Algorithmic Thinking

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Why Graphs?

- ❖ Biological networks

- ❖ Maps

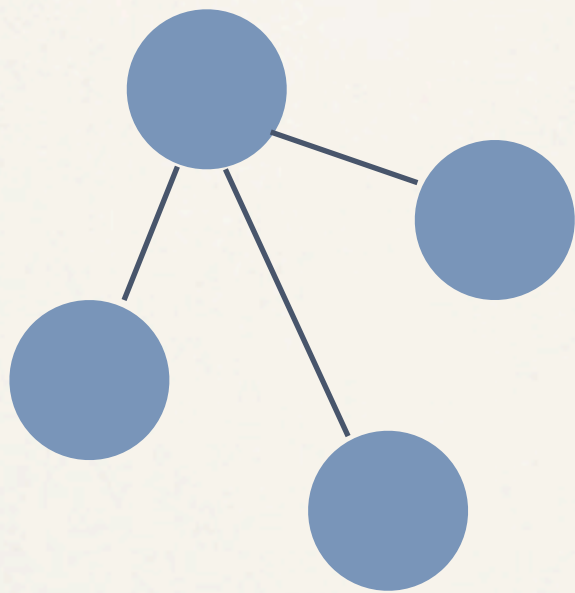
- ❖ Social networks

- ❖ ...

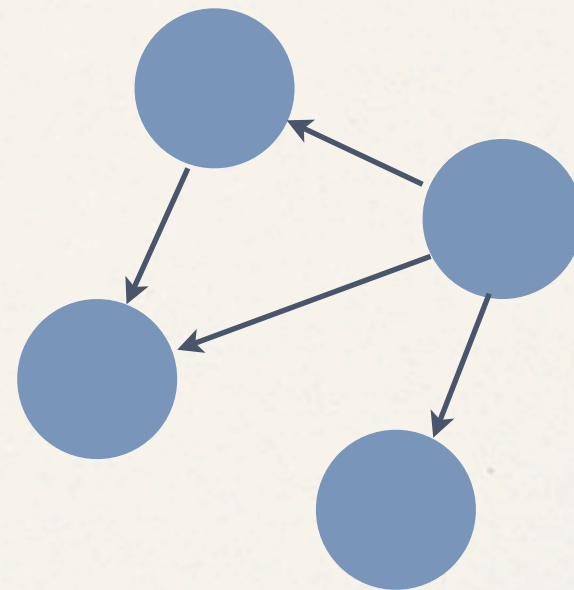
Graphs: Basic Definitions

The Connectivity of Networks

- ❖ Common to all networks we saw is that the connectivity of each of them can be represented via some form of a graph



Undirected graph



Directed graph

1 : node
(or, vertex)

1 — 2 : edge
 $\{1,2\}$

1 → 2 : directed edge
 $(1,2)$

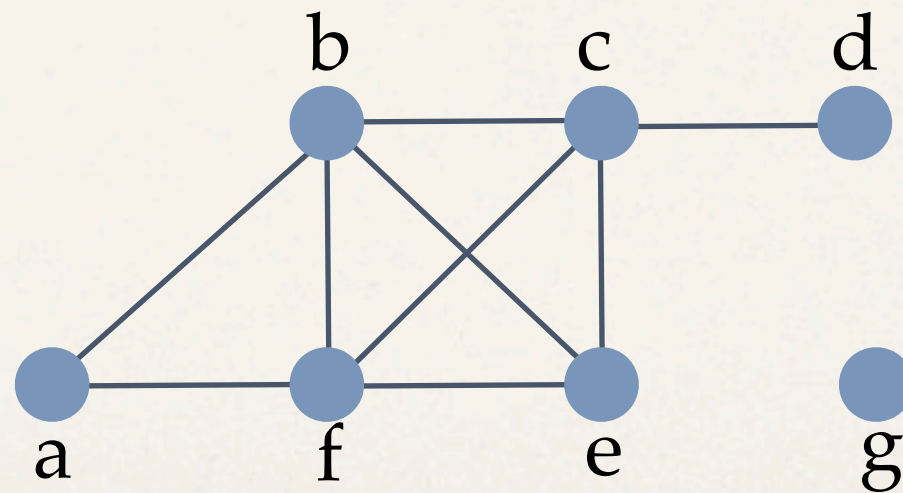
Graphs

- ❖ An **undirected graph**, or **graph** for short, G , is a pair (V, E) , where
 - ❖ $V = \{0, 1, \dots, n-1\}$ is a nonempty set of nodes, and
 - ❖ $E \subseteq \{\{i, j\} : i, j \in V\}$ is a set of unordered pairs, each of which corresponds to an undirected edge in the graph G .
- ❖ A **directed graph**, or **digraph** for short, G , is a pair (V, E) , where
 - ❖ $V = \{0, 1, \dots, n-1\}$ is a nonempty set of nodes, and
 - ❖ $E \subseteq (V \times V)$ is a set of ordered pairs, each of which corresponds to a directed edge in the graph G .

IMPORTANT: In this course, graphs have no self-loops or parallel edges, unless explicitly stated otherwise.

Basic Terminology

- ❖ Two nodes i and j in a graph $G=(V,E)$ are called adjacent (or neighbors) in G if there is an edge between i and j ; that is, if $\{i,j\}\in E$.
- ❖ The degree of a node in an undirected graph is the number of edges incident with it. The degree of node i is denoted by $\deg(i)$.

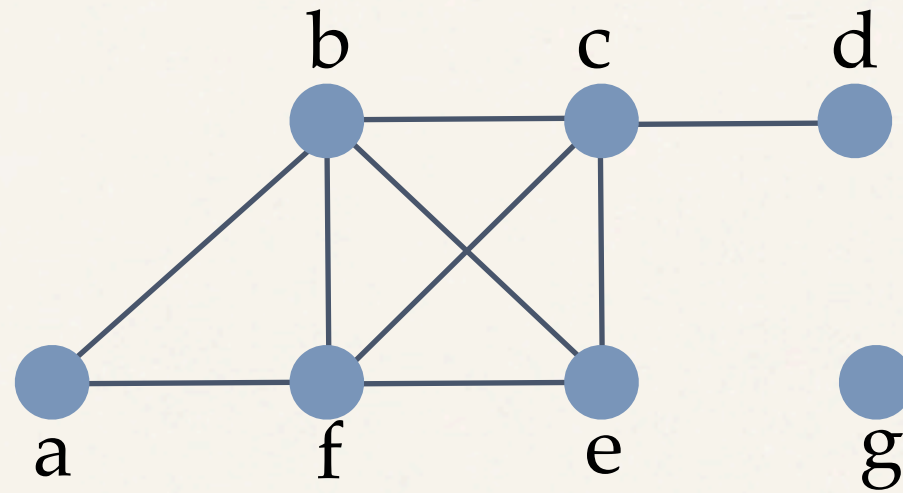


What are the degrees of the nodes?

Degree Distribution

- ✧ Define p_k to be the fraction of nodes in the graph that have degree k .
- ✧ The degree distribution of a graph can be visualized by making a histogram of the p_k values.

Degree Distribution



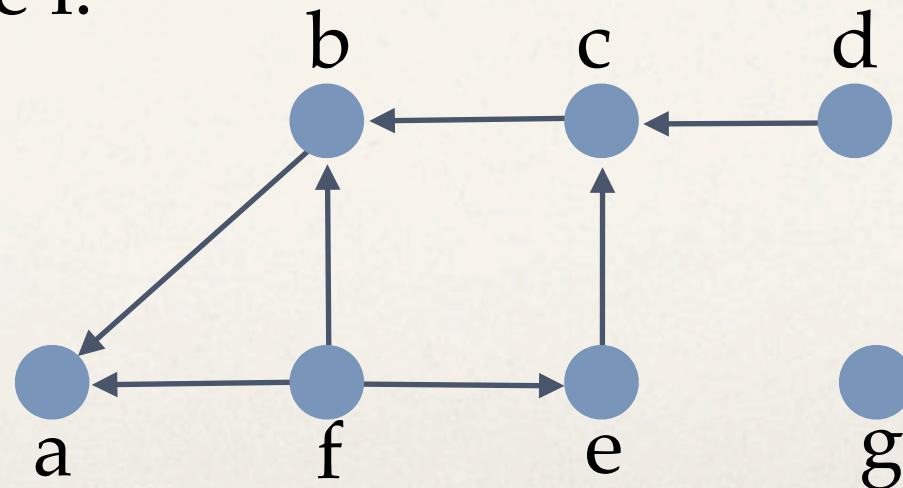
$$\begin{aligned} p_0 &= 1/7 & p_1 &= 1/7 & p_2 &= 1/7 \\ p_3 &= 1/7 & p_4 &= 3/7 \end{aligned}$$

Notice that if m is the highest node degree, then:

$$\sum_{k=0}^m p_k = 1$$

Basic Terminology

- ❖ If $e=(i,j)$ is a directed edge from node i to node j , we say that i is the tail (or, initial node) of e and j is the head (or, terminal node) of e .
- ❖ The in-degree of a node i in a directed graph is the number of edges whose head is the node i . The in-degree of node i is denoted by $\text{indeg}(i)$.
- ❖ The out-degree of i , denoted by $\text{outdeg}(i)$, is the number of edges whose tail is the node i .



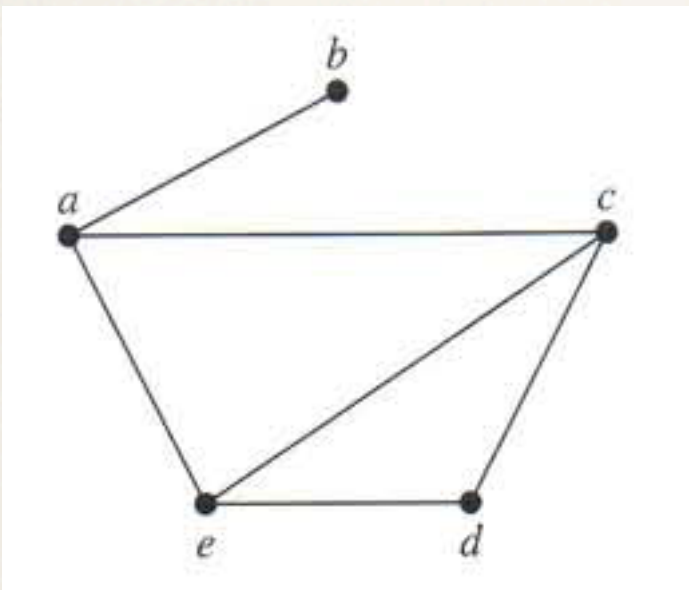
What are the in- and out-degrees of the nodes?

In- and Out-Degree Distributions

- ❖ For in-degree distribution: Define p_k to be the fraction of nodes in the graph that have in-degree k .
- ❖ For out-degree distribution: Define q_k to be the fraction of nodes in the graph that have out-degree k .
- ❖ The in- and out-degree distributions of a graph can be visualized by making a histogram of the p_k and q_k values, respectively.

Graph Representation: Adjacency Lists

- ❖ Adjacency lists specify the nodes that are adjacent to each node of the graph.

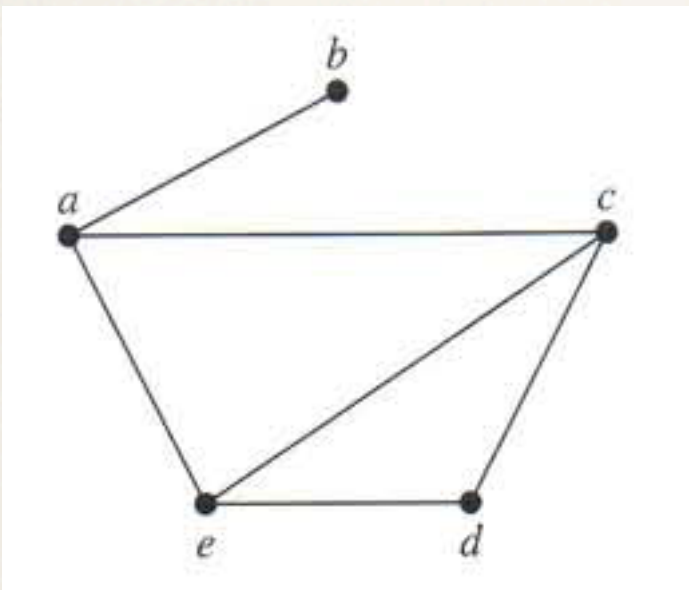


Node	Adjacent nodes
<i>a</i>	<i>b, c, e</i>
<i>b</i>	<i>a</i>
<i>c</i>	<i>a, d, e</i>
<i>d</i>	<i>c, e</i>
<i>e</i>	<i>a, c, d</i>

While “adjacency list” is a historical name, the adjacent nodes of a given node form a set; so, you can think of this representation as an “adjacency set.”

Graph Representation: Adjacency Lists

- Adjacency lists specify the nodes that are adjacent to each node of the graph.



Node	Adjacent nodes
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

Notice the redundancy!
(ensures symmetry)
In the case of digraphs,
there is no redundancy.

While “adjacency list” is a historical name, the adjacent nodes of a given node form a set; so, you can think of this representation as an “adjacency set.”

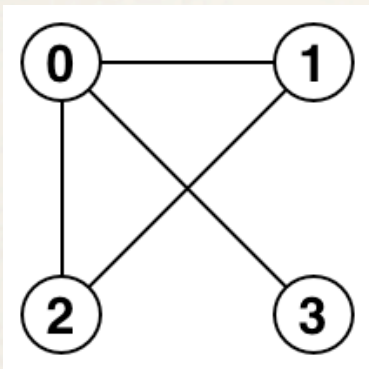
Adjacency Lists and Their Dictionary Representation

Node	Adjacent nodes
<i>a</i>	<i>b, c, e</i>
<i>b</i>	<i>a</i>
<i>c</i>	<i>a, d, e</i>
<i>d</i>	<i>c, e</i>
<i>e</i>	<i>a, c, d</i>

{a: set([b,c,e]),
b: set([a]),
c: set([a,d,e]),
d: set([c,e]),
e: set([a,c,d])}

Graph Representation: Adjacency Matrices

- ❖ Let $G=(V,E)$ be a graph with $V=\{0,1,\dots,n-1\}$
- ❖ The adjacency matrix of G , denoted by A_G , is the $n \times n$ 0-1 matrix with 1 as its $(i,j)^{\text{th}}$ entry when i and j are adjacent, and 0 as its $(i,j)^{\text{th}}$ entry when i and j are not adjacent.



G

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

A_G

Notice the redundancy!

In the case of digraphs, A_G is not necessarily symmetric.

Draw a graph whose
adjacency matrix is

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Trade-offs Between Adjacency Lists and Adjacency Matrices

- ❖ When the graph is sparse, i.e., contains relatively few edges (relative to what?), it is usually preferable to use adjacency lists (Why?)
- ❖ When the graph is dense, i.e., contains relatively many edges (again, relative to what?), it is usually preferable to use adjacency matrices (Why?)

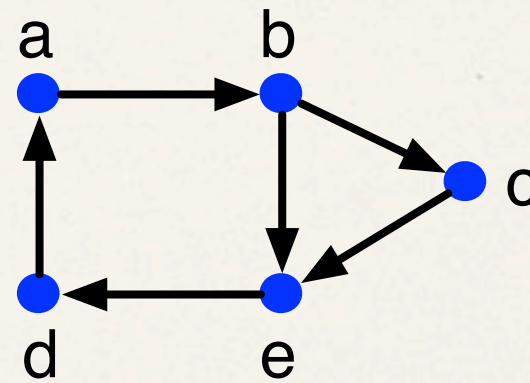
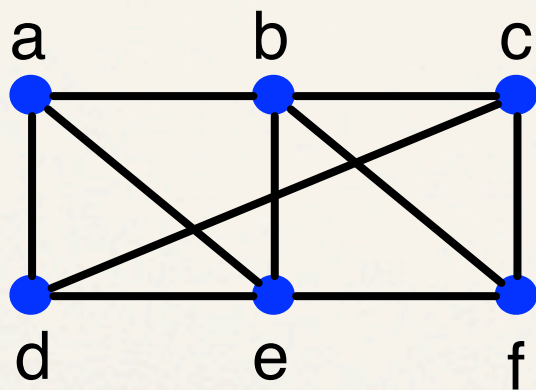
Graph Connectivity: Paths

- ❖ Let k be a nonnegative integer and G a graph.
 - ❖ A path of length k from node v_0 to node v_k in G is a sequence of k edges e_1, e_2, \dots, e_k of G such that $e_1 = \{v_0, v_1\}$, $e_2 = \{v_1, v_2\}$, ..., $e_k = \{v_{k-1}, v_k\}$, where v_0, \dots, v_k are all nodes in V , and e_1, \dots, e_k are all edges in E .
- ❖ We usually denote such a path by its node sequence (v_0, v_1, \dots, v_k) .
- ❖ A path is simple if it does not contain the same node more than once.
- ❖ A cycle is a simple path that begins and ends at the same node.
- ❖ A path (not necessarily simple) that begins and ends at the same node is called a circuit.

Graph Connectivity: Paths

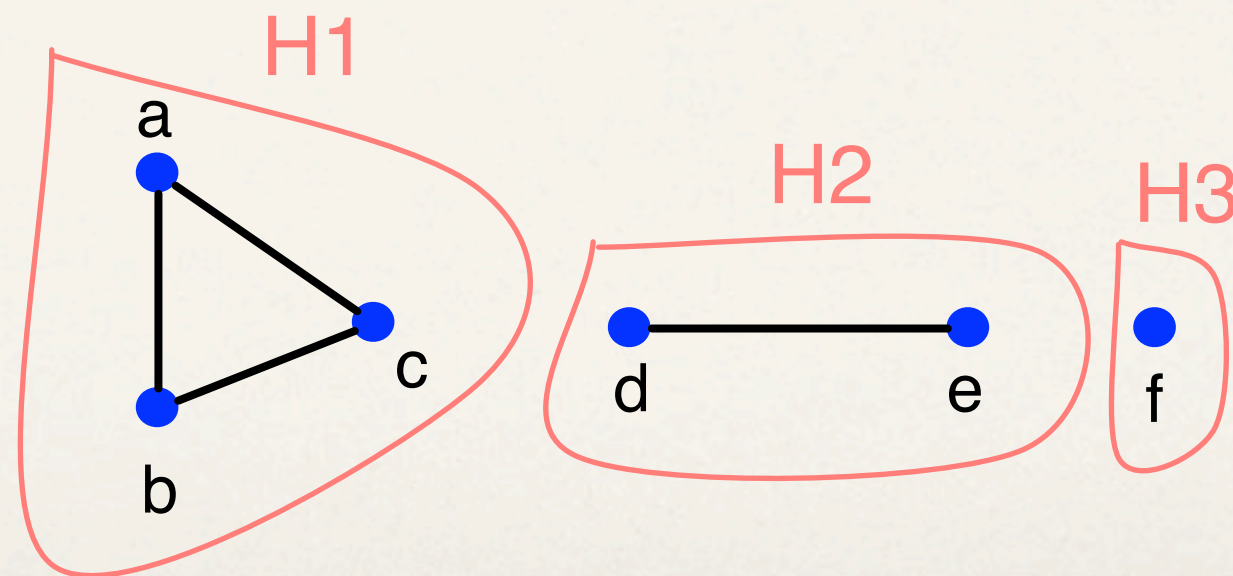
- ❖ If G is a directed graph, a path must traverse edges in their respective directions.

Show a simple path from node e to node d in each of the following two graphs:



Graph Connectivity

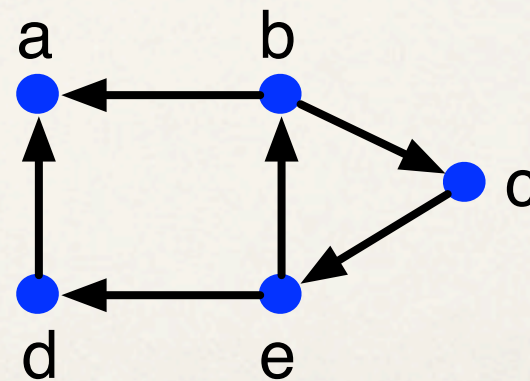
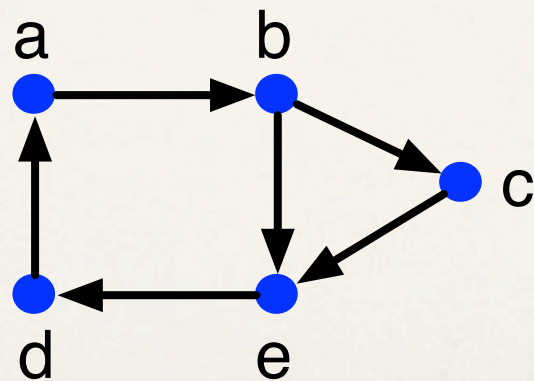
- ❖ An undirected graph is called connected if there is a path between every pair of distinct nodes of the graph.
- ❖ A connected component (CC) of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G .



A graph with 3 CCs

Graph Connectivity

- ❖ A digraph is strongly connected if there is a path from i to j for every pair of nodes i and j of the digraph (note that there must be a path from i to j and another from j to i).
- ❖ A digraph is weakly connected if there is a path between every two nodes in the underlying undirected graph.
- ❖ The subgraphs of a directed graph G that are strongly connected but not contained in larger strongly connected subgraphs are called the strongly connected components (SCC) of G .



Strongly connected? Weakly connected? What are the SCCs?